Trigonometric Functions

The class of trigonometric functions is an important class of continuous functions. Here we study some of the common properties of these functions.

x	$\sin x$	$\cos x$	$\tan x$	$\cot x$	$\sec x$	$\csc x$
0	0	1	0	$\pm\infty$	1	$\pm\infty$
$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$	$\sqrt{3}$	$2\sqrt{3}/3$	2
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1	1	$\sqrt{2}$	$\sqrt{2}$
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$	$\sqrt{3}/3$	2	$2\sqrt{3}/3$
$\pi/2$	1	0	$\pm\infty$	0	$+\infty$	1
$-x$	$-\sin x$	$\cos x$	$-\tan x$	$-\cot x$	$\sec x$	$-\csc x$
$\pi/2-x$	$\cos x$	$\sin x$	$\cot x$	$\tan x$	$\csc x$	$\sec x$
$\pi/2+x$	$\cos x$	$-\sin x$	$-\cot x$	$-\tan x$	$-\csc x$	$\sec x$
$\pi-x$	$\sin x$	$-\cos x$	$-\tan x$	$-\cot x$	$-\sec x$	$\csc x$
$\pi+x$	$-\sin x$	$-\cos x$	$\tan x$	$\cot x$	$-\sec x$	$-\csc x$

Identities.

$\sin^2 x + \cos^2 x = 1$ $\quad$ $\sec^2 x - \tan^2 x = 1$ $\quad$ $\csc^2 x - \cot^2 x = 1$

$\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a$ $\quad$ $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$ $\quad$ $\cot(a \pm b) = \frac{1 \pm \cot a \cot b}{\cot a \mp \cot b}$

$\cos a \cos b = \frac{\cos(a+b)+\cos(a-b)}{2}$ $\quad$ $\sin a \sin b = \frac{\cos(a-b)-\cos(a+b)}{2}$

$\sin a \cos b = \frac{\sin(a+b)+\sin(a-b)}{2}$ $\quad$ $\sin a \cos a = \frac{\sin 2a}{2}$

$\sin^2 a = \frac{1-\cos 2a}{2}$ $\quad$ $\cos^2 a = \frac{1+\cos 2a}{2}$

Limits

$\lim_{x\to 0} \frac{\sin x}{x} = 1$ $\quad$ $\lim_{x\to\infty} \frac{\sin x}{x} = 0$

Mac Lauren Series

$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ $\quad$ $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

Complex forms

$\sin x = \frac{e^{ix}-e^{-ix}}{2i}$ $\quad$ $\cos x = \frac{e^{ix}+e^{-ix}}{2}$

de Moivre's formula

$(\cos x + i \sin x)^n = \cos nx + i \sin nx$

Let $T = (a, b, c; A, B, C)$ be a triangle , where a, b, c are the sides and A, B, C are the angles. Then T satisfies the following conditions:

Law of Sines	$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$
Law of Cosines	$c^2 = a^2 + b^2 - 2ab \cos C$

$f(x)$	$\sin x$	$\cos x$	$\tan x$	$\cot\ x$	$\sec x$	$\csc\ x$
$\sin x$	$\sin x$	$\sqrt{1-\cos^2 x}$	$\frac{\tan x}{\sqrt{1+\tan^2 x}}$	$\frac{1}{\sqrt{1+\cot^2 x}}$	$\frac{\sqrt{\sec^2 x-1}}{\sec x}$	$\frac{1}{\csc x}$
$\cos x$	$\sqrt{1-\sin^2 x}$	$\cos x$	$\frac{1}{\sqrt{1+\tan^2 x}}$	$\frac{\cot x}{\sqrt{1+\cot^2 x}}$	$\frac{1}{\sec x}$	$\frac{\sqrt{\csc^2 x-1}}{\csc x}$
$\tan x$	$\frac{\sin x}{\sqrt{1-\sin^2 x}}$	$\frac{\sqrt{1-\cos^2 x}}{\cos x}$	$\tan x$	$\frac{1}{\cot x}$	$\sqrt{\sec^2 x-1}$	$\frac{1}{\sqrt{\csc^2 x-1}}$
$\cot x$	$\frac{\sqrt{1-\sin^2 x}}{\sin x}$	$\frac{\cos x}{\sqrt{1-\cos^2 x}}$	$\frac{1}{\tan x}$	$\cot x$	$\frac{1}{\sqrt{\sec^2 x-1}}$	$\sqrt{\csc^2 x-1}$
$\sec x$	$\frac{1}{\sqrt{1-\sin^2 x}}$	$\frac{1}{\cos x}$	$\sqrt{1+\tan^2 x}$	$\frac{\cot x}{\sqrt{1+\cot^2 x}}$	$\sec x$	$\frac{\csc x}{\sqrt{\csc^2 x-1}}$
$\csc x$	$1/\sin x$	$1/\sqrt{1-\cos^2 x}$	$\frac{\sqrt{1+\tan^2 x}}{\tan x}$	$\sqrt{1+\cot^2 x}$	$\frac{\sec x}{\sqrt{\sec^2 x-1}}$	$\csc x$

Derivatives and Integrals.

$f(x)$	$\sin x$	$\cos x$	$\tan x$	$\cot\ x$	$\sec x$	$\csc\ x$
$f'(x)$	$\cos x$	$-\sin x$	$sec^2\ x$	$-\csc^2 x$	$\sec x\tan x$	$-\csc x\cot x$
$\int f(x)dx$	$-\cos x$	$\sin x$	$\ln\lvert\sec x\rvert$	$\ln\lvert\sin x\rvert$	$\ln\lvert\sec x+\tan x\rvert$	$\ln\lvert\csc x-\cot x\rvert$

Inverse Trigonometric Functions.

Let $f(x) = \sin x$, then the inverse function $f^{-1}(x) = arcsin\ x = \sin^{-1} x$ is called the arcsine of x. If $f(x)$ is one to one in a domain, then $f^{-1}(x)$ is also a function. Thus $\sin(arcsin\ x) = x$ for $x \in [-1, 1]$ and $arcsin\,(\sin x) = x$ whenever $x \in [-\pi/2, \pi/2]$. Similarly we may define $arccos\ x$, $arctan\ x$, etc...

$f(x)$	$arcsin\ x$	$arccos\ x$	$arctan\ x$	$arccot\ x$	$arcsec\ x$	$arccsc\ x$
Domain	$[-1,1]$	$[-1,1]$	$(-\infty,\infty)$	$(-\infty,\infty)$	$\lvert x\rvert\geq 1$	$\lvert x\rvert\geq 1$
Range	$\lvert x\rvert\leq\frac{\pi}{2}$	$[0,\pi]$	$\lvert x\rvert\leq\frac{\pi}{2}$	$(0,\pi)$	$[0,\pi]-\{\frac{\pi}{2}\}$	$\lvert x\rvert\leq\frac{\pi}{2}$
$f'(x)$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{-1}{\sqrt{1-x^2}}$	$\frac{1}{1+x^2}$	$\frac{-1}{1+x^2}$	$\frac{1}{x\sqrt{x^2-1}}$	$\frac{-1}{x\sqrt{x^2-1}}$

To integrate inverse trigonometric functions, we use integration by parts with $v(x) = 1$.

$$\int \sin^{-1} x\ dx = x\ \sin^{-1} x + \sqrt{1-x^2} + C \qquad \int \cos^{-1} x\ dx = x\ \cos^{-1} x - \sqrt{1-x^2} + C$$

$$\int \tan^{-1} x\ dx = x\ \tan^{-1} x - \tfrac{1}{2}\ln(1+x^2) + C \qquad \int \cot^{-1} x\ dx = x\ \cot^{-1} x + \tfrac{1}{2}\ln(1+x^2) + C$$

$$\int \sec^{-1} x\, dx = x \sec^{-1} x - \ln\left|x + \sqrt{x^2-1}\,\right| + C \qquad \int \csc^{-1} x\ dx = x\ \csc^{-1} x + \ln\left|x + \sqrt{x^2-1}\,\right| + C$$

Identities.

$$arcsin\ x + arccos\ x = \pi/2 \quad and \quad arcsin\ x = arccsc\ \left(\tfrac{1}{x}\right), \quad for\ |x| \leq 1.$$

$$arcsec\ x + arccsc\ x = \pi/2 \quad and \quad arcsec\ x = arccos\ \left(\tfrac{1}{x}\right), \quad for\ |x| \geq 1.$$

$$cos(arcsin\ x) = \sin(arccos\ x) = \sqrt{1-x^2} \qquad \sec(arctan\ x) = \sqrt{x^2+1}$$

$$arcsin\ (-x) = -\arcsin x \quad arctan\ (-x) = -arctan\ x \quad arccos\ (-x) = \pi - arccos\ x$$