Inverse of Lower Triangular Matrices

Given an $n \times n$ matrix $M = (m_{ij})$, we will use the following notations:

$$M = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{pmatrix} = \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{bmatrix} = [\, M^1 \quad M^2 \quad \cdots \quad M^n \,] .$$

We illustrate the algorithm of finding the inverse of a lower triangular matrix, where all the diagonal entries are equal to one, by the use of the following 6×6 matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & 1 & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & 1 & 0 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & 1 & 0 \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 3 & 4 & 1 & 0 & 0 & 0 \\ 5 & 6 & -1 & 1 & 0 & 0 \\ 8 & 9 & -10 & 5 & 1 & 0 \\ -2 & -4 & 6 & 5 & -3 & 1 \end{pmatrix}.$$

In order to find A^{-1} we proceed as follows:

Step 1. The matrix $A^{-1} = B = (b_{ij})$ is obtained from A by first choosing

$[b_{21} = -a_{21} = -2]$ $[b_{32} = -a_{32} = -4]$ $[b_{43} = -a_{43} = 1]$ $[b_{54} = -a_{54} = -5]$
$[\ b_{65} = -a_{65} = 3]$;

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 \\ b_{31} & -4 & 1 & 0 & 0 & 0 \\ b_{41} & b_{42} & 1 & 1 & 0 & 0 \\ b_{51} & b_{52} & b_{53} & -5 & 1 & 0 \\ b_{61} & b_{62} & b_{63} & b_{64} & 3 & 1 \end{pmatrix}.$$

Step 2. Now, we find

$[b_{31} = a_{21} \times a_{32} - a_{31} = 8 - 3 = 5]$ $[b_{42} = a_{32} \times a_{43} - a_{42} = -4 - 6 = -10]$
$[b_{53} = a_{43} \times a_{54} - a_{53} = -5 - (-10) = 5]$ $[b_{64} = a_{54} \times a_{65} - a_{64} = -15 - 5 = -20]$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 \\ 5 & -4 & 1 & 0 & 0 & 0 \\ b_{41} & -10 & 1 & 1 & 0 & 0 \\ b_{51} & b_{52} & 5 & -5 & 1 & 0 \\ b_{61} & b_{62} & b_{63} & -20 & 3 & 1 \end{pmatrix}$$

Step 3. Next we find b_{41}, b_{52}, and b_{63}. Note that $BA = I_n$, hence

$B_4A^1 = b_{41} \times 1 + (-10) \times 2 + 1 \times 3 + 1 \times 5 = b_{41} - 12 = 0 \Rightarrow b_{41} = 12;$
$B_5A^2 = b_{52} \times 1 + 5 \times 4 + (-5) \times 6 + 1 \times 9 = b_{52} - 1 = 0 \Rightarrow b_{52} = 1;$
$B_6A^3 = b_{63} \times 1 + (-20) \times (-1) + 3 \times (-10) + 1 \times 6 = b_{63} - 4 = 0 \Rightarrow b_{63} = 4.$

Thus

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 \\ 5 & -4 & 1 & 0 & 0 & 0 \\ 12 & -10 & 1 & 1 & 0 & 0 \\ b_{51} & 1 & 5 & -5 & 1 & 0 \\ b_{61} & b_{62} & 4 & -20 & 3 & 1 \end{pmatrix}.$$

Step 4. Next we set B_5A^1 and B_6A^2 to zero and obtain:

$$B_5A^1 = b_{51} \times 1 + 1 \times 2 + 5 \times 3 + (-5) \times 5 + 1 \times 8 = b_{51} - 0 = 0 \Rightarrow b_{51} = 0;$$
$$B_6A^2 = b_{62} \times 1 + 4 \times 4 + (-20) \times 6 + 3 \times 9 + 1 \times (-4) = b_{62} - 81 = 0 \Rightarrow b_{62} = 81.$$

Hence

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 \\ 5 & -4 & 1 & 0 & 0 & 0 \\ 12 & -10 & 1 & 1 & 0 & 0 \\ 0 & 1 & 5 & -5 & 1 & 0 \\ b_{61} & 81 & 4 & -20 & 3 & 1 \end{pmatrix}.$$

Step 5. Finally by setting B_6A^1 to zero, we obtain b_{61}, the last unknown entry of $A^{-1} = B$

$$B_6A^1 = b_{61} \times 1 + 81 \times 2 + 4 \times 3 + (-20) \times 5 + 3 \times 8 + 1 \times (-2) = b_{61} + 96 = 0 \Rightarrow b_{61} = -96.$$

Hence

$$A^{-1} = B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 \\ 5 & -4 & 1 & 0 & 0 & 0 \\ 12 & -10 & 1 & 1 & 0 & 0 \\ 0 & 1 & 5 & -5 & 1 & 0 \\ -96 & 81 & 4 & -20 & 3 & 1 \end{pmatrix}.$$

Remark 1. If the invertible matrix A has some diagonal entries different from one, then we first define the matrix

$$C = DA = \operatorname{diag}\left(\frac{1}{a_{11}}, \frac{1}{a_{22}}, \cdots, \frac{1}{a_{nn}}\right) A$$

which has ones on its main diagonal. From the identity $C^{-1} = A^{-1}D^{-1}$, we obtain

$$A^{-1} = C^{-1}D = C^{-1}\operatorname{diag}\left(\frac{1}{a_{11}}, \frac{1}{a_{22}}, \cdots, \frac{1}{a_{nn}}\right).$$

Remark 2. In order to find the inverse of an upper triangular matrix A, we first transpose the matrix to change it into a lower triangular matrix, then we apply our algorithm to A^t. Once $(A^t)^{-1} = (A^{-1})^t$ is found, we only need to transpose it in order to obtain A^{-1}.