Matrix Exponential

Let A be an $n \times n$ real or complex matrix. We denote the $n \times n$ identity matrix by I_n and zero matrix by Z_n. The exponential of A, denoted by e^A or $exp(A)$, is the $n \times n$ matrix given by the power series

$$e^A = I_n + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots + \frac{1}{k!}A^k + \cdots\cdots = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

The above series always converges, so the exponential of A is well-defined.

Properties. *Let A and B be $n \times n$ real or complex matrices and let λ and μ be some scalers. The matrix exponential satisfies the following properties:*

1. $e^{Z_n} = I_n$.
2. $e^{\lambda A} e^{\mu B} = e^{(\lambda+\mu)A}$.
3. *If* $AB = BA$, *then* $e^A e^B = e^{A+B}$.
4. $e^A e^{-A} = I_n$.
5. *A commutes with e^{tA}, that is,* $Ae^{tA} = e^{tA}A$.
6. *If* $D = diag\,(d_1, d_2, \ldots, d_i, \ldots, d_n)$, *then* $e^D = diag\left(e^{d_1}, e^{d_2}, \ldots, e^{d_i}, \ldots, e^{d_n}\right)$.
7. *If B is invertible then* $e^{BAB^{-1}} = Be^AB^{-1}$.
8. $\det(e^A) = e^{tr(A)}$.
9. $e^{A^t} = (e^A)^t$. *It follows that if A is symmetric then e^A is also symmetric, and that if A is skew-symmetric then e^A is orthogonal (i.e.,* $e^A(e^A)^t = I_n$*).*
10. $e^{A^*} = (e^A)^*$, *where A^* denotes the conjugate transpose of A. It follows that if A is Hermitian (i.e., $A = A^*$) then e^A is also Hermitian, and that if A is skew-Hermitian then e^A is unitary (i.e.,* $e^A(e^A)^* = I_n$.

Let A be an $n \times n$ matrix and J its Jordan canonical form (i.e., $A = PJP^{-1}$, for some invertible matrix P); then $J = D + N$, where D is a diagonal matrix and N a nilpotent matrix, that is $A^k = Z_n$ for some positive integer k. Since D commutes with N, we have

$$e^A = e^{PJP^{-1}} = Pe^JP^{-1} = Pe^{D+N}P^{-1} = Pe^De^NP^{-1}.$$

Example. Let $A = \begin{pmatrix} -9 & -4 & 6 & 9 \\ -56 & -20 & 33 & 45 \\ 4 & 2 & -1 & -3 \\ -41 & -16 & 24 & 35 \end{pmatrix}$ with eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = 2$ and $\lambda_4 = -1$. Then the Jordan canonical form

$$J = D + N = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

is obtain from:

$$P^{-1}AP = \begin{pmatrix} 14 & 5 & -7 & -11 \\ -11 & -4 & 6 & 9 \\ -9 & -3 & 5 & 7 \\ 5 & 2 & -3 & -4 \end{pmatrix} \begin{pmatrix} -9 & -4 & 6 & 9 \\ -56 & -20 & 33 & 45 \\ 4 & 2 & -1 & -3 \\ -41 & -16 & 24 & 35 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 1 & 1 & 2 & 3 \\ 1 & 0 & 1 & -1 \\ 1 & 3 & -1 & 2 \end{pmatrix}.$$

We have

$$e^J = e^{D+N} = e^D e^N = \begin{pmatrix} e^2 & 0 & 0 & 0 \\ 0 & e^2 & 0 & 0 \\ 0 & 0 & e^2 & 0 \\ 0 & 0 & 0 & e^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^2 & e^2 & 0 & 0 \\ 0 & e^2 & 0 & 0 \\ 0 & 0 & e^2 & 0 \\ 0 & 0 & 0 & e^{-1} \end{pmatrix}.$$

Hence

$$\begin{aligned} e^A = PJP^{-1} &= \begin{pmatrix} 1 & 2 & -1 & 0 \\ 1 & 1 & 2 & 3 \\ 1 & 0 & 1 & -1 \\ 1 & 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} e^2 & e^2 & 0 & 0 \\ 0 & e^2 & 0 & 0 \\ 0 & 0 & e^2 & 0 \\ 0 & 0 & 0 & e^{-1} \end{pmatrix} \begin{pmatrix} 14 & 5 & -7 & -11 \\ -11 & -4 & 6 & 9 \\ -9 & -3 & 5 & 7 \\ 5 & 2 & -3 & -4 \end{pmatrix} \\ &= \begin{pmatrix} -10e^2 & -4e^2 & 6e^2 & 9e^2 \\ -26e^2 + 15e^{-1} & -9e^2 + 6e^{-1} & 15e^2 - 9e^{-1} & 21e^2 - 12e^{-1} \\ -6e^2 - 5e^{-1} & -2e^2 - 2e^{-1} & 4e^2 + 3e^{-1} & 5e^2 + 4e^{-1} \\ -21e^2 + 10e^{-1} & -8e^2 + 4e^{-1} & 12e^2 - 6e^{-1} & 18e^2 - 8e^{-1} \end{pmatrix}. \end{aligned}$$

♣ **System of Linear Differential Equations**. One of the reasons for the importance of the matrix exponential is that it can be used to solve systems of linear ordinary differential equations

$$X'(t) = AX(t) + b(t),$$

where A is a constant matrix.

If we can calculate e^{tA}, then we can obtain the solution to the system. By making e^{tA} an integrating factor and multiplying throughout, we obtain

$$\begin{aligned} & e^{-tA}X'(t) - e^{-tA}AX(t) = e^{-tA}b(t) \\ & \frac{d}{dt}\left[e^{-tA}X(t)\right] = e^{-tA}b(t) \\ & X(t) = e^{tA}\left[\int_0^t e^{-uA}b(u)\,du\right]. \end{aligned}$$

There is no closed-form solution, where A is not a constant matrix.

Given an initial value Problem.

$$\begin{cases} X'(t) = AX(t) \\ X(0) = X_0, \end{cases}$$

where A is not a constant matrix. The solution is given by

$$X(t) = e^{At}X_0.$$

Example 1. Solve the linear homogeneous system:

$$\begin{cases} x' &= 2x & -y & +z \\ y' &= & 3y & -z \\ z' &= 2x & +y & +3z \end{cases}$$

We have the associated matrix $A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{pmatrix}$. First we find the Jordan normal form as follows:

$$J = P^{-1}AP = \begin{pmatrix} 1/2 & 0 & 1/2 \\ -1/2 & 0 & 1/2 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

$$e^{tA} = P\begin{pmatrix} e^{4t} & 0 & 0 \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{pmatrix}P^{-1} = \frac{1}{2}\begin{pmatrix} e^{4t} + e^{2t} - 2te^{2t} & -2te^{2t} & e^{4t} - e^{2t} \\ -e^{4t} - e^{2t} + 2te^{2t} & 2te^{2t} & -e^{4t} + e^{2t} \\ e^{4t} - e^{2t} + 2te^{2t} & 2te^{2t} & e^{4t} + e^{2t} \end{pmatrix}$$

Hence
$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = C_1 \begin{bmatrix} e^{4t} + e^{2t} - 2te^{2t} \\ -e^{4t} - e^{2t} + 2te^{2t} \\ e^{4t} - e^{2t} + 2te^{2t} \end{bmatrix} + C_2 \begin{bmatrix} -2te^{2t} \\ 2te^{2t} \\ 2te^{2t} \end{bmatrix} + C_3 \begin{bmatrix} e^{4t} - e^{2t} \\ -e^{4t} + e^{2t} \\ e^{4t} + e^{2t} \end{bmatrix}.$$

Example 2. Solve the linear homogeneous system:

$$X'(t) = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} X(t).$$

All the eigenvalues of A are equal to 1; so we can find $exp(A)$ without using the Jordan normal form. From the facts that

$$A - I_4 = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (A - I_4)^2 = Z_4, \quad A = (A - I_4) + I_4\,,$$

and $t(A - I_4)$ commutes with tI_4, we have

$$e^{tA} = e^{t[A-I_4]+tI_4} = e^{t[A-I_4]}e^{tI_4} = \begin{pmatrix} 1 & t & -t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^t & 0 & 0 \\ 0 & 0 & e^t & 0 \\ 0 & 0 & 0 & e^t \end{pmatrix} = \begin{pmatrix} e^t & te^t & -te^t & 0 \\ 0 & e^t & 0 & te^t \\ 0 & 0 & e^t & te^t \\ 0 & 0 & 0 & e^t \end{pmatrix}.$$

Thus
$$X(t) = \begin{bmatrix} C_1e^t + C_2te^t - C_3te^t \\ C_2e^t + C_4te^t \\ C_3e^t + C_4te^t \\ C_4e^t \end{bmatrix}.$$