Direct Factorization of Matrices

We discuss the factoring of a square matrix A in terms of a lower-triangular matrix L and an upper-triangular matrix U. It is known that this factorization exists whenever the linear system $Ax = b$ can be solved uniquely by Gaussian elimination without row interchanges. The system $LUx = Ax = b$ can then be transformed into the system $Ux = L^{-1}b$ and, since U is upper triangular, backward substitution can be applied.

♣ **Doolittle's Decomposition**. This method allows us to factor a square matrix A into LU, where L is a lower-triangular matrix with ones on the main diagonal and U an upper-triangular matrix with nonzero diagonal entries.

ALGORITHM.

Input a square matrix A.
$n = size(A);\ m = n(1);$ (Find the size of A.)
$U = zeros(m);$ (Define U as an $m \times m$ zero matrix.)
$L = eye(m);$ (Define L as an $m \times m$ identity matrix.)
$U(1, 1:m) = A(1, 1:m),\ L(2:m, 1) = (1/U(1, 1)) * A(2:m, 1)$
$for\ k = 2:m$
  $for\ j = k:m,\ U(k, j) = A(k, j) - L(k, 1:k-1) * U(1:k-1, j);\ end;\ U$
  $for\ i = k+1:m,\ L(i, k) = (A(i, k) - L(i, 1:k) * U(1:k, k))/U(k, k);\ end;\ L$
end

Example. Consider the matrix $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 5 & 6 & 8 & 8 \\ 1 & 3 & 6 & 5 \end{pmatrix} = \begin{bmatrix} A(1, 1:4) \\ A(2, 1:4) \\ A(3, 1:4) \\ A(4, 1:4) \end{bmatrix} = [A(i, j)].$

First, we initialize U and L.

$$U = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The first row of U is : $U(1, 1:4) = A(1, 1:4)$.
The first column of L is : $L(2:4, 1) = (1/U(1, 1)) * A(2:4, 1)$

$$U = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 5 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

The second row of U is : $for\ j = 2:4,\ U(2, j) = A(2, j) - L(2, 1:1) * U(1:1, j);\ end.$
The second column of L is: $for\ i = 3:4,\ L(i, 2) = (A(i, 2) - L(i, 1:2) * U(1:2, 2))/U(2, 2);\ end.$

$$U = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 5 & 4 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}$$

The third row of U is : $for\ j = 3:4,\ U(3,j) = A(3,j) - L(3,1:2) * U(1:2,j);\ end.$
The third column of L is: $for\ i = 3:4,\ L(i,3) = (A(i,3) - L(i,1:3) * U(1:3,3))/U(3,3);\ end.$

$$U = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 5 & 4 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

The last row of U is : $for\ j = 4:4,\ U(4,j) = A(4,j) - L(4,1:3) * U(1:3,j);\ end.$
The last column of L is: $for\ i = 4:4,\ L(i,4) = (A(i,4) - L(i,1:4) * U(1:4,4))/U(4,4);\ end.$

$$U = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 5 & 4 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

♣ **Choleski's Decomposition**. If A is a positive definite symmetric matrix , then A has a factorization of the form LL^t, where L is a lower-triangular matrix. This factorization is known as the Choleski's decomposition method.

ALGORITHM.

Input a positive definite symmetric matrix A.
$n = size(A);\ m = n(1);$ % Find the size of A.
$D = zeros(m);$ % Define the diagonal matrix D as an $m \times m$ zero matrix.
$L = zeros(m);$ % Define L as an $m \times m$ zero matrix.
$D(1,1) = sqrt(A(1,1)),\ L(2:m,1) = (1/D(1,1)) * A(2:m,1)$
$fork = 2:m$
  $for\ j = k:m,\ D(k,k) = sqrt(A(k,k) - L(k,1:k) * L(k,1:k)^t);\ end,\ D$
  $for\ i = k+1:m,\ L(i,k) = (A(i,k) - L(i,1:k) * L(k,1:k)^t)/D(k,k);\ end;\ L$
end
$D(m,m) = sqrt(A(m,m) - L(m,1:m) * L(m,1:m)^t)$
$L = L + D$

Example. Consider the positive definite symmetric matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 13 & 18 & 23 \\ 3 & 18 & 50 & 62 \\ 4 & 23 & 62 & 126 \end{pmatrix} = \begin{bmatrix} A(1,1:4) \\ A(2,1:4) \\ A(3,1:4) \\ A(4,1:4) \end{bmatrix} = [A(i,j)].$$

We initialize D and L.

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first diagonal entry of D is : $D(1,1) = \sqrt{A(1,1)}$.
The first column of L is : $L(2:4,1) = (1/D(1,1)) * A(2:4,1)$

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}$$

The second diagonal entry of D is : $D(2,2) = \sqrt{A(2,2) - L(2,1:1) * L(2,1:1)^t}$.
The second column of L is: $for\ i = 3:4,\ L(i,2) = (A(i,2) - L(i,1:2)*L(2,1:2))/D(2,2);\ end.$

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 4 & 5 & 0 & 0 \end{bmatrix}$$

The third diagonal entry of D is : $D(3,3) = \sqrt{A(3,3) - L(3,1:2) * L(3,1:2)^t)}$.
The third column of L is: $for\ i = 3:4,\ L(i,3) = (A(i,3) - L(i,1:3) * (3,1:3)^t)/D(3,3);\ end.$

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 4 & 5 & 6 & 0 \end{bmatrix}$$

The last diagonal entry of D is : $D(4,4) = \sqrt{A(4,4) - L(4,1:3) * L(4,1:3)^t}$.
The matrix $L = L + D$

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 3 & 4 & 5 & 0 \\ 4 & 5 & 6 & 7 \end{bmatrix}$$

♣ **Inverse of Bidiagonal Matrices**. Consider a $n \times n$ non singular lower-triangular matrix A where all the diagonals are zero, except the main diagonal D, and the L diagonal below that. The following algorithm explains how the inverse of A is obtained.

ALGORITHM.

Input the diagonal D of a matrix A.
Input its lower-diagonal L.
$n = size(D);\ m = n(2);$ % Find the size of A.
$A = zeros(m);\ B = eye(m);\ C = A;$ % Initialize A, B and C.
$for\ k = 1:m,\ A(k,k) = D(k);\ C(k,k) = (1/D(k));\ end;$
$for\ k = 2:m,\ A(k,k-1) = L(k-1);\ B(k,k-1) = -(1/D(k-1)) * L(k-1);\ end;\ A,\ B$
$for\ j = 2:m-1;$
  $for\ k = j+1:m,\ B(k,k-j) = B(k-1,k-j) * B(k,k-1);\ end;\ B$
end
The inverse of A is $B = C * B$

Example. Consider the matrix

$$A = \begin{pmatrix} 4 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & -5 & 4 \end{pmatrix} = \begin{bmatrix} A(1,1:4) \\ A(2,1:4) \\ A(3,1:4) \\ A(4,1:4) \end{bmatrix} = [A(i,j)].$$

We have:

$$D = \begin{bmatrix} 4 \\ 3 \\ -1 \\ 4 \end{bmatrix} \qquad L = \begin{bmatrix} -2 \\ -3 \\ -5 \end{bmatrix} \qquad C = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$for\ \ k = 2:m,\ \ B(k,k-1) = -(1/D(k-1)) * L(k-1);\ \ end;\ \ A,\ \ B$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -5 & 1 \end{bmatrix}$$

$for\ \ k = 3:4,\ \ B(k,k-2) = B(k-1,k-2) * B(k,k-1);\ \ end;\ \ B$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 0 & -5 & -5 & 1 \end{bmatrix}$$

$for\ \ k = 4:4,\ \ B(k,k-3) = B(k-1,k-3) * B(k,k-1);\ \ end;\ \ B$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 10 & -5 & -5 & 1 \end{bmatrix}$$

The inverse of A is $B = C * B$

$$A^{-1} = B = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 2 & -1 & -1 & 0 \\ \frac{5}{2} & -\frac{5}{4} & -\frac{5}{4} & \frac{1}{4} \end{bmatrix}$$

♣ **Crout's Reduction For Tridiagonal Linear Systems**. This is an algorithm for solving an $n \times n$ system of linear equations whose coefficient matrix is tridiagonal.

Consider the linear system $Ax = b$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & \vdots \\ \vdots & a_{32} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & a_{n-1,n} \\ 0 & \cdots & \cdots & a_{n,n-1} & a_{nn} \end{pmatrix}$$

is a non singular tridiagonal matrix.

First we obtain $A = LU$, where

$$L = \begin{pmatrix} l_{11} & 0 & \cdots & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & \vdots \\ \vdots & l_{32} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & l_{n,n-1} & l_{nn} \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 & u_{12} & \cdots & \cdots & 0 \\ 0 & 1 & u_{23} & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & u_{n-1,n} \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix},$$

then we solve the system $Lz = b$, and finally we obtain x by solving $Ux = z$.

ALGORITHM.

Input the tridiagonal matrix A and the vector b.
$n = size(A); m = n(1);$ % Find the size of A.
$L = zeros(m);$ % Define L as an $m \times m$ zero matrix.
$U = eye(m);$ % Define U as an $m \times m$ identity matrix.
$z = zeros(m,1);\ \ x = zeros(m,1)$ % z and x are zero vectors.
$for\ \ i = 2 : m,\ \ L(i,i-1) = A(i,i-1);\ \ end;$
$L(1,1) = A(1,1),\ \ U(1,2) = (1/L(1,1)) * A(1,2)$
$for\ \ k = 2 : m,$
  $for\ \ j = k : m,$
    $L(k,k) = A(k,k) - L(k,k-1) * U(k-1,k);$
    $U(k,k+1) = A(k,k+1)/L(k,k);$
  $end;\ \ L,\ \ U$
end
$L(m,m) = A(m,m) - L(m,m-1) * U(m-1,m);\ \ L$
$z(1) = b(1)/L(1,1);$
$for\ \ k = 2 : m,\ \ z(k) = (1/L(k,k)) * (b(k) - L(k,k-1) * z(k-1));\ \ end;\ \ z$
$x(m) = z(m);$
$for\ \ k = m-1 : -1 : 1,\ \ x(k) = z(k) - U(k,k+1) * x(k+1);\ \ end;\ \ x$
end

Example. Consider the linear system $Ax = b$, where

$$A = \begin{pmatrix} 2 & -2 & 0 & 0 \\ 1 & 2 & -6 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 3 & -4 \end{pmatrix} = \begin{bmatrix} A(1,1:4) \\ A(2,1:4) \\ A(3,1:4) \\ A(4,1:4) \end{bmatrix} = [A(i,j)] \qquad b = \begin{pmatrix} 2 \\ 4 \\ -10 \\ -4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

We initialize L, U, x and z.

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad z = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We have :
$for\ \ i = 2 : m,\ \ L(i,i-1) = A(i,i-1);\ \ end;$
$L(1,1) = A(1,1),\ \ U(1,2) = (1/L(1,1)) * A(1,2)$

$$L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$L(2,2) = A(2,2) - L(2,1) * U(1,2);$
$U(2,3) = A(2,3)/L(2,2);$

$$L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$L(3,3) = A(3,3) - L(3,2) * U(2,3);$
$U(3,4) = A(3,4)/L(3,3);$

$$L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$L(4,4) = A(4,4) - L(4,3) * U(3,4);$

$$L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$z(1) = b(1)/L(1,1);$
$for\ \ k = 2 : m,\ \ z(k) = (1/L(k,k)) * (b(k) - L(k,k-1) * z(k-1));\ \ end;\ \ z$

$$z = L^{-1}b = \begin{bmatrix} -38 \\ -40 \\ -22 \\ -4 \end{bmatrix}$$

$x(m) = z(m);$
$for\ \ k = m-1 : -1 : 1,\ \ x(k) = z(k) - U(k,k+1) * x(k+1);\ \ end;\ \ x$
end

$$x = U^{-1}z = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$