

Linear Differential Equations

♣ **Linearly Independent functions.** The set \mathbf{F} of all complex-valued functions is known to be a vector space of infinite dimension. A set $\{f_k(x)\}$ ($k = 1, 2, \dots, n$) of functions is said to be *linear independent*, if $f_k(x)$'s satisfy the following condition:

$$c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0 \quad \text{if and only if} \quad c_1 = c_2 = \dots = c_n = 0.$$

Let $f_1(x), f_2(x), \dots, f_n(x)$ be $n - 1$ times differentiable functions. The function

$$W[f_1(x), f_2(x), \dots, f_n(x)] = \det \begin{pmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ f_1'(x) & f_2'(x) & \dots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{pmatrix}$$

is called the *Wronskian* of $f_k(x)$'s.

To prove that $\{f_k(x)\}$ ($k = 1, 2, \dots, n$) is a linearly independent set, it is usually simpler to show that

$$W(f_1(x), f_2(x), \dots, f_n(x)) \neq 0.$$

For example $\sin x$ and $\cos x$ are linearly independent since

$$W(\cos x, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0.$$

Theorem 1. Consider the n -th order homogeneous linear differential equation

$$[a_0(x)D^n + a_1(x)D^{n-1} + \dots + a_{n-1}(x)D + a_n(x)I] y = 0,$$

where the coefficients $a_i(x)$'s are continuous functions. Then the solutions to this equation form a vector space of dimension n . Thus if $f_1(x), f_2(x), \dots, f_n(x)$ are solutions to the equation and

$$W[f_1(x), f_2(x), \dots, f_n(x)] \neq 0,$$

then

$$\sum_{k=1}^n c_k f_k(x) \quad (c_k \in \mathbf{C})$$

is also a solution.

Theorem 2. Let $f(x)$ be a solution to the equation $P(x, D)y = 0$ and let $g(x)$ a solution to the equation $Q(x, D)y = 0$. Then $af(x) + bg(x)$ is a solution to the equation $P(x, D)Q(x, D)y = 0$.

The general solution of the n -th order non-homogeneous linear differential equation $P(x, D)y = R(x)$ is $y = y_c + y_p$, where y_c , called the *complementary solution* is a solution to the homogeneous equation $P(x, D)y = 0$, and y_p is a *particular solution* to $P(x, D)y = R(x)$.

If $P(x, D)y = R_1(x) + R_2(x) + \dots + R_s(x)$, then $y_p = y_{p_1} + y_{p_2} + \dots + y_{p_s}$, where each y_{p_k} is a particular solution to the equation $P(x, D)y = R_k(x)$.

♣ **Linear Differential Equations With Constant Coefficients.** Recall that

$\cosh x = \frac{e^x + e^{-x}}{2}$	$\sinh x = \frac{e^x - e^{-x}}{2}$
$\cos x = \frac{e^{ix} + e^{-ix}}{2}$	$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

By using the fact that $y = e^{ax}$ is a solution to $(D - aI)y = 0$, we obtain

Equation:	Solution
$[(D - aI)^2 - b^2]y = 0$	$y = c_1 \cosh x + c_2 \sinh x$
$[(D - aI)^2 + b^2]y = 0$	$y = c_1 \cos x + c_2 \sin x$

If y is a solution to $[(D - aI)^2 \pm b^2]y = 0$, then $x^k y$ ($k = 0, 1, \dots, n - 1$) are linearly independent solutions of $[(D - aI)^2 \pm b^2]^n y = 0$

♡ **The Method of Undetermined Coefficients.** let $P(D)y = R(x)$, where

$$P(D) = a_0 D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n I.$$

To solve the equation, we proceed as follows

Step 1. we solve the homogeneous equation $P(D)y = 0$ and find y_c .

Step 2. We find an equation $F(D)y = 0$ which has y_p as a solution.

Step 3. We solve the homogeneous equation $P(D)F(D)y = 0$ and find the general solution $y = y_c + y_p$. Here y_p has undetermined coefficients.

Step 4. Finally, we find the undetermined coefficients of y_p by using the linear system $P(D)y_p = R(x)$.

Equation:	$[D^2 - 2D - 3I]y = 10 \sin x$
Step 1.	$y_c = c_1 e^{3x} + c_2 e^{-x}$
Step 2.	$F(D) = [D^2 + 1]$
Step 3.	$y = c_1 e^{3x} + c_2 e^{-x} + A \cos x + B \sin x$
Step 4.	$y_p = \cos x - 2 \sin x$

♡ **Exponential Shift.** If $P(D)y = R(x)$, then

$$e^{-ax} P(D)y = P(D + a)(e^{-ax} y) = e^{-ax} R(x).$$

To solve this new equation, we set $z = e^{-ax} y$, then we solve the equation $P(D + a)z = e^{-ax} R(x)$. The solution to the original equation will be

$$y = e^{ax} z = e^{ax} z_c + e^{ax} z_p.$$

Consider the polynomial:

$$P(D) = b_0 D^n + b_1 D^{n-1} + \dots + b_{n-k} D^k \quad (0 \leq k \leq n).$$

If for some a , $P(D - aI) = (D - I)^r Q(D)$, where $Q(a) \neq 0$, then

Equation:	Solution
$P(D)y = c$	$y_p = \frac{cx^k}{k!b_n - k}$
$P(D)y = ce^{ax}$	$y_p = \frac{cx^r e^{ax}}{r!Q(a)}$

♣ **Second Order Equations.** The exponential shift may be used to solve the equation

$$[a_0D^2 + a_1D + a_2I]y = R(x).$$

Algorithm. Any second order Linear differential equation with constant coefficients may be expressed as

$$[(D - aI)(D - bI)]y = R(x).$$

Step 1. Multiply both sides of the equation by e^{-ax} .

$$\begin{aligned} e^{-ax}[(D - aI)(D - bI)]y &= e^{-ax}R(x) \\ D[D + (a - b)I](e^{-ax}y) &= e^{-ax}R(x) \end{aligned}$$

Step 2. Set $z_1 = e^{-ax}$ and then integrate both sides of the equation.

$$[D + (a - b)I]z_1 = \int e^{-ax}R(x) = R_1(x)$$

Step 3. Use the first step, the new equation, and $e^{(a-b)x}$.

$$\begin{aligned} e^{(a-b)x}[D + (a - b)I]z_1 &= e^{(a-b)x}R_1(x) \\ [D + (a - b)I](e^{(a-b)x}z_1) &= e^{(a-b)x}R_1(x) \end{aligned}$$

Step 4. To obtain the solution, set $z_2 = e^{(a-b)x}z_1 = e^{-bx}y$ and then integrate.

$$z_2 = e^{-bx}y = \int e^{-ax}R(x) = R_2(x)$$

♠ **Note.** Clearly we can use this method for higher order equation, but it is not always possible to find roots of a polynomial of degree higher than 2.

Here are some examples.

$$\text{If } [D^2 + I]y = R(x) \text{ then } y_p = \left(-\int R(x) \sin x dx\right) \cos x + \left(\int R(x) \cos x dx\right) \sin x$$

$$\begin{aligned} \text{If } [D^2 + a^2I]y = c \cos ax \text{ then } y_p &= -\frac{cx}{2a} \cos ax \\ \text{If } [D^2 + a^2I]y = c \sin ax \text{ then } y_p &= \frac{cx}{2a} \sin ax \end{aligned}$$

For $|a| \neq |b|$, we have

If $[D^2 \pm a^2 I]y = c \cos bx$ then $y_p = -\frac{c \cos bx}{\pm a^2 - b^2}$
If $[D^2 \pm a^2 I]y = c \sin bx$ then $y_p = \frac{c \sin bx}{\pm a^2 - b^2}$

♡ **Reduction of Order.** Consider the second order homogeneous differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0.$$

If y_1 is one of the solutions of this equation, then we can always express the second solution as $y_2 = v(x)y_1$ for some function $v(x)$. So our task is to find the function $v(x)$.

In order to find $v(x)$, first we find

$$y_2' = y_1'v(x) + y_1v'(x) \quad y_2'' = y_1''v(x) + 2y_1'v'(x) + y_1v''(x),$$

then we set y_2 in the equation and obtain

$$a_0(x)[y_1''v(x) + 2y_1'v'(x) + y_1v''(x)] + a_1(x)[y_1'v(x) + y_1v'(x)] + a_2(x)[y_1v(x)] = 0.$$

Next we set $w(x) = v'(x)$. Since y_1 is a solution to our homogeneous equation i.e.,

$$a_0(x)y_1'' + a_1(x)y_1' + a_2(x)y_1 = 0,$$

the above equation reduces to the following *first order linear differential equation*

$$a_0(x)[2y_1'w(x) + y_1w'(x)] + a_1(x)[y_1w(x)] = 0.$$

By solving this equation we find $w(x)$. The function $v(x)$ can be then obtained by integrating $w(x)$.

Equation:	$(x^2 + 1)y'' - 2xy' + 2y = 0$
Given	$y_1 = x$
Step 1.	$y_2 = xv(x) \quad y' = xv'(x) - v(x) \quad y'' = xv''(x) - 2v'(x)$
Step 2.	$w(x) = v'(x) \quad x(x^2 + 1)w'(x) + 2w(x) = 0$
Step 3.	$w(x) = \frac{x^2+1}{x^2} \quad v(x) = x - \frac{1}{x} \quad y_2 = xv(x) = x^2 - 1$
solution	$y = C_1x + C_2(x^2 - 1)$

♡ **Variation Of Parameters.** Let $y'' + P(x)y' + Q(x)y = R(x)$ and let $y_p = A(x)y_1 + B(x)y_2$, where y_1 and y_2 are the complementary solutions. We assume that $A'(x)y_1 + B'(x)y_2 = 0$ and solve the linear system

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} A'(x) \\ B'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ R(x) \end{bmatrix}$$

$A(x)$ and $B(x)$ can be obtained by integrating $A(x)$ and $B(x)$.

Equation:	$y'' + y' = \tan x$
Step 1.	$y_1 = \cos x \quad y_2 = \sin x$
Step 2.	$y'_1 = -\sin x \quad y'_2 = \cos x$
Step 3.	$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} A'(x) \\ B'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \tan x \end{bmatrix}$
Step 4.	$A'(x) = \sin x \quad B'(x) = \cos x - \sec x$
Step 5.	$A(x) = -\cos x \quad B(x) = \sin x - \ln \sec x + \tan x $

♣ **Linear Differential Equations With Non-Constant Coefficients.** In the preceding section we showed how to obtain the general solution, using complementary solution. It is usually difficult to find complementary solution of a linear differential equation with non-constant coefficients even when the order is only 2. But sometimes one can use a transformation which changes the equation with non-constant coefficients into an equation with constant coefficients.

One special class of equations with non-constant coefficients, for which the complementary solution can be found is the so-called class of *Cauchy-Euler* equations.

♣ **The Cauchy-Euler Equation.** The equation

$$[a_0x^n D_x^n + a_1x^{n-1} D_x^{n-1} + \dots + a_{n-1}x D_x + a_n I]y = R(x)$$

is known as the *Cauchy-Euler* equation. By setting $x = e^t$ in the equation, we obtain an equation with constant coefficient.

Theorem 3. *If $x = e^t$, then $t = \ln x$ and*

$$D_x^n y = \frac{1}{x^n} \left[\prod_{k=0}^{n-1} (D_t - kI) \right] y.$$

By using the above theorem, we change the Cauchy-Euler equation into

$$\left[a_0 \left[\prod_{k=0}^{n-1} (D_t - kI) \right] + a_1 \left[\prod_{k=0}^{n-2} (D_t - kI) \right] + \dots + a_{n-1} D_t + a_n I \right] y = R(e^t).$$

This equation has only constant coefficients.

Equation:	$[x^3 D_x^3 - 4x^2 D_x^2 + 8x D_x - 8I] y = \ln x$
Step 1.	$x = e^t \quad t = \ln x$
Step 2.	$D_x = \frac{1}{x} D_t \quad D_x^2 = \frac{1}{x^2} D_t(D_t - I) \quad D_x^3 = \frac{1}{x^3} D_t(D_t - I)(D_t - 2I)$
Equation:	$[D_t^3 - 7D_t^2 + 14D_t - 8I] y = 4t$