Laplace Transforms

A real function $F(t)$ is said to be of *Class A*, if $F(t)$ is *Sectionally Continuous* for $t > 0$, and

$$\lim_{t\to\infty} e^{-t^r} F(t) = 0 \quad (r > 1).$$

If $F(t)$ is a function of *Class A*, then for some s, we define the *Laplace Transform* of $F(t)$ denoted by $L_s\{F(t)\}$ as follows:

$$L_s\{F(t)\} = \int_0^\infty e^{-st} F(t) dt.$$

$$\boxed{\begin{array}{lll} (1) & s > a & L_s\{F(t)e^{at}\} = L_{s-a}\{F(t)\} \\ (2) & n \in I\!N & L_s\{t^n\} = \frac{n!}{s^{n+1}} \\ (3) & n = 2p+1 & L_s\{t^{\frac{n}{2}}\} = \frac{1}{s^{p+1}} \left(\frac{n}{2} \frac{n-2}{2} \ldots \frac{3}{2} \frac{1}{2} \sqrt{\frac{\pi}{s}} \right) \end{array}}$$

By differentiating N times $F(t)$ and then by taking the Laplace Transform, we have:

$$\boxed{(4) \qquad L_s\{F^{(N)}(t)\} = s^N L_s\{F(t)\} - s^{N-1}F(0) - s^{N-2}F'(0) \ldots F^{N-1)}(0)}$$

By differentiating N times the Laplace Transform of $F(t)$ with respect to the parameter s, we obtain

$$\boxed{(5) \qquad \frac{d^N}{ds^N} \left(L_s\{F(t)\} \right) = L_s\{(-t)^N F(t)\}}$$

An immediate consequence of the above formula is:

$$\boxed{(6) \qquad L_s\{t^N F(t)\} = (-1)^N \frac{d^N}{ds^N} \left(L_s\{F(t)\} \right)}$$

By integrating the Laplace Transform of $F(t)$, we obtain:

$$\boxed{(7) \qquad L_s\left\{\frac{F(t)}{t}\right\} = \int_s^\infty L_r\{F(t)\} dr}$$

A function $F(t)$ is called *Periodic of Period* ω , if for all t in the domain of $F(t)$,

$$F(t + \omega) = F(t).$$

$$\boxed{(8) \qquad F(t+\omega) = F(t) \qquad L_s\{F(t)\} = \tfrac{1}{1-e^{-s\omega}} \int_0^{\omega} e^{-su} F(u)du}$$

♣ **Gamma Function**. For $x > 0$, we define the Gamma function as follows:

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt = L_s\{t^{x-1}\}.$$

$$\boxed{\begin{array}{lll} (9) & s > 0 & \Gamma(x+1) = x\Gamma(x) = s^{x+1} L_s\{t^x\} \\ (10) & n \in I\!N & \Gamma(n+1) = n\Gamma(n) = n! \\ (11) & & \Gamma(\tfrac{1}{2}) = \sqrt{\pi} \end{array}}$$

♣ **A Step Function**. We define the step function $\alpha(t)$ as follows:

$$\alpha(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{otherwise.} \end{cases}$$

Thus

$$\alpha(t-a) = \begin{cases} 0, & \text{if } t < a \\ 1, & \text{otherwise;} \end{cases} \qquad \alpha(t-a)F(t) = \begin{cases} 0, & \text{if } t < a \\ \text{F(t)}, & \text{otherwise;} \end{cases}$$

and

$$[\alpha(t-a) - \alpha(t-b)]F(t) = \begin{cases} \text{F(t)}, & \text{if } a < t < b\,; \\ 0, & \text{otherwise.} \end{cases}$$

By using the α function, we obtain:

$$\boxed{(12) \quad L_s\{\alpha(t-a)F(t-a)\} = e^{-as} L_s\{F(t)}$$

Inverse Transform

The Laplace Transform is a *one to one* function, therefore its *inverse function* is also a function and is called the *Inverse Transform* denoted by $L_t^{-1}\{f(s)\}$; so

$$F(t) = L_t^{-1}\left\{L_s\{F(t)\}\right\}.$$

♣ **The Convolution Formula**. If $f(s) = L_s\{F(t)\}$ and $g(s) = L_s\{G(t)\}$, then

$$\boxed{(13) \quad L_t^{-1}\{f(s)g(s)\} = \int_0^t F(u)G(t-u)du = \int_0^t F(t-u)G(u)du}$$

Since $L_s\{\alpha(t)\} = \frac{1}{s}$, it follows that

$$\boxed{(14) \quad L_t^{-1}\left\{\tfrac{f(s)}{s}\right\} = \int_0^t F(u)du}$$

♣ **Trigonometric and Hyperbolic Functions**. Finally we give some useful formulas for the inverse transform.

$$\begin{array}{ll}
L_t^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin(kt) & L_t^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt) \\
L_t^{-1}\left\{\frac{k}{s^2-k^2}\right\} = \sinh(kt) & L_t^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh(kt) \\
L_t^{-1}\left\{\arctan(\frac{k}{s})\right\} = \frac{\sin(kt)}{t} & L_t^{-1}\left\{\ln\left(\frac{s+k}{s-k}\right)\right\} = \frac{2\sinh(kt)}{t}
\end{array}$$

♣ **Laplace Transform Solution of Linear Differential Equations** We now consider how the Laplace Transform may be applied to solve the initial-value problem consisting of the nth-order linear differential equation with constant coefficients.

Consider the initial-value problem:

$$\begin{cases} a_0 y(n) + a_1 y^{(} n-1) + \cdots + a_{n-1} y' + a_n y = b \\ \quad y(0) = c_0, y'(0) = c_1, \ldots, y^{(n-1)}(0) = c_{n-1}. \end{cases}$$

By taking first, the Laplace Transform of both sides of the equation and then the inverse Transform of the resulting equation, we may find a solution to the problem. We illustrate our method with the following example.

Solve	$y'' - 6y' + 9y = t^2e^3t$
subject to	$y(0) = 2, \quad y'(0) = 6.$
Step 1.	$L_s\{y''\} - L_s\{y'\} + 9L_s\{y\} = L_s\{t^2e^3t\}$
Step 2.	$Y(s) = \frac{2}{s-3} + \frac{2}{(s-3)^5}$
Step 3.	$y(t) = L_t^{-1}\{Y(s)\} = 2e^3t + \frac{1}{12}t^4e^3t$

Finally, we find the solution of a system of linear differential equations with initial conditions.

Solve	$\begin{cases} x' - 6x + 3y = 8e^t, \\ y' - 2x - y = 4e^t, \end{cases}$
subject to	$x(0) = -1, \quad y(0) = 0.$
Step 1.	$\begin{cases} L_s\{x'\} - 6L_s\{x\} + 3L_s\{y\} = L_s\{8e^t\} \\ L_s\{y'\} - 2L_s\{x\} - L_s\{y\} = L_s\{4e^t\} \end{cases}$
Step 2.	$\begin{cases} X(s) = \dfrac{-s+7}{(s-1)(s-4)} \\ Y(s) = \dfrac{2}{(s-1)(s-4)} \end{cases}$
Step 3.	$\begin{cases} x(t) = L_t^{-1}\{X(s)\} = -2e^t + e^{4t} \\ y(t) = L_t^{-1}\{Y(s)\} = \dfrac{1}{3}(-2e^t + 2e^{4t}) \end{cases}$